

ON MAJORANA'S EQUATION

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ABSTRACT. The physical results of quantum field theory are *independent* of the various specializations of Dirac's gamma-matrices, that are employed in given problems. Accordingly, the physical meaning of Majorana's equation is very dubious, considering that it is a consequence of *ad hoc* matrix representations of the gamma-operators. Therefore, it seems to us that this equation cannot give the equation of motion of the neutral WIMPs (weakly interacting massive particles), the hypothesized constitutive elements of the Dark Matter.

Key words: SUSY-particles; Dark Matter.

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1. – Let us consider the Dirac equation for the field operator ψ in the absence of any interaction with other fields:

$$(1) \quad i \gamma^\mu \frac{\partial \psi}{\partial x^\mu} - m \psi = 0 \quad , \quad (\hbar = c = 1; \quad \mu = 0, 1, 2, 3) \quad ;$$

$$(1') \quad \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \eta^{\mu\nu} \quad ,$$

where $\eta^{\mu\nu}$ is the customary Minkowskian tensor.

The corresponding charge-conjugate operator ψ_C satisfies the *same* Dirac equation (1) [1].

(We recall that with any matrix representation of the γ^μ 's, the operator ψ_C is a simple function of the adjoint of ψ).

It is clear that Dirac equation (1) describes perfectly also the *neutral* particles with their *identical* antiparticles.

2. – In quantum field theory the physical concepts (*e.g.*, the energy eigenvalues, the four-current j^μ , *etc.*) and the physical conclusions are *independent* of any particular matrix representation of Dirac's γ^μ -operators. As it is known.

3. – Majorana’s equation (see [2] and [3]) is a specialization of Dirac equation [1] such that the elements of the γ^μ -matrices are all *imaginary*. Consequently, the expression $[i\gamma_{im}^\mu \partial_\mu - m]$ is a *real* expression, and we can put, with Majorana, $\psi = \varphi + i\chi$, with a “real” φ and a “real” χ – two *selfadjoint* operators –, that satisfy the *same* Dirac equation. And it seems that Majorana’s equation

$$(2) \quad i\gamma_{im}^\mu \frac{\partial \varphi}{\partial x^\mu} - m\varphi = 0$$

describes particles which coincide with their antiparticles, *i.e.* neutral objects.

However, this conclusion depends on a particular choice of the γ^μ -matrices, and is consequently very problematic. Moreover eq. (2) is *not* invariant under the phase (*gauge*) transformations of φ :

$$(3) \quad \begin{cases} \varphi(x) \rightarrow \varphi'(x) \equiv \varphi(x) \exp[\pm iF(x)] & , \\ \frac{\partial}{\partial x^\mu} \rightarrow \frac{\partial}{\partial x^\mu} \mp \frac{\partial F(x)}{\partial x^\mu} & . \end{cases}$$

Of course, if we write

$$(4) \quad i\gamma_{im}^\mu \frac{\partial \psi}{\partial x^\mu} - m\psi = 0 \quad , \quad (\psi = \varphi + i\chi) \quad ,$$

we have a standard instance of Dirac equation (1), which is invariant under the phase transformations of ψ .

4. – According to many astrophysicists, Dark Matter is composed of WIMPs (weakly interacting massive particles) [4]. And there is a widespread belief that these neutral particles are described by Majorana’s equation (2). Now, the above considerations show that this conviction is not well founded.

The motions of the hypothetical WIMPs can be properly described by means of Dirac equation (1).

APPENDIX A

The condition $\gamma^\mu = \gamma_{im}^\mu$, ($\mu = 0, 1, 2, 3$), is necessary and sufficient for the formal validity of Majorana’s equation (2). Indeed, let us assume that the γ^μ ’s are different from the γ_{im}^μ ’s, but – for simplicity – with the same transposition on properties of the γ_{im}^μ ’s *i.e.*: $\gamma^{0T} = -\gamma^0$; $\gamma^{kT} = \gamma^k$, ($k = 1, 2, 3$) [3]. We know that, quite generally, γ^0 is Hermitian and the γ^k ’s are anti-Hermitian.

Denoting the adjointness with an asterisk, from the equation

$$(5) \quad \left(i \gamma^0 \partial_0 + i \gamma^k \partial_k - m \right) \psi = 0$$

we get easily:

$$(6) \quad \left(i \gamma^0 \partial_0 + i \gamma^k \partial_k - m \right) \psi^{*T} = 0 \quad .$$

If ψ_C is the charge-conjugate operator of ψ , we have:

$$(7) \quad \left(i \gamma^0 \partial_0 + i \gamma^k \partial_k - m \right) \psi_C = 0 \quad .$$

Now, the assumption

$$(8) \quad \psi_C = \psi^{*T}$$

shows that eq. (7) coincides with eq. (6); and we see that Majorana's condition that ψ is selfadjoint:

$$(9) \quad \psi_C = \psi^{*T} = \psi$$

implies $\gamma^\mu = \gamma_{im}^\mu$, *i.e.* the assumption that the expression (...) of eq. (5) is *real*.

Remak that the restriction $\gamma^\mu = \gamma_{im}^\mu$ *without* the assumption (8) tells us that $\psi = \varphi + i\chi$, with φ and χ selfadjoint operators.

It is now obvious to conclude for the validity of Majorana's equation (2). It is not difficult to generalize the above reasoning for any transposition property of $\gamma^0, \gamma^1, \gamma^2, \gamma^3$.

APPENDIX B

We have considered Majorana's equation, with Majorana [2] and Wilczek [3], from the standpoint of quantum field theory ("second" quantization of Dirac theory). Since *ubi maius minus cessat*, the previous treatment implies that a consideration of Majorana's equation as a mathematical object of a "first" quantization, or of a classical field theory (wave-picture, *Wellenbild*, in Heisenberg's terminology) is quite superfluous. However, we think that is is useful to emphasize the following points.

- i)* "First" quantization: the wave-function ψ of Schrödinger and Dirac equations must be *complex* (see Pauli [5]) – and this fact is sufficient to discard Majorana's equation.
- ii)* Classical field theory: the probability density and the probability current-density of the "first" quantization assumes a realistic meaning of matter density and matter current-density, which involves obviously a *non-real* classical field ψ : a Majorana's equation does not make sense.

APPENDIX C

We have emphasized in sect. 1 that a Dirac neutral particle coincides with its antiparticle. For a Weyl neutral particle things go otherwise [6]. Indeed, the charge conjugate of Weyl equation describes an antiparticle, which is characterized by the opposite sign of particle *helicity* $\vec{\sigma} \cdot \vec{p}/|\vec{p}|$. This unique difference between particle and antiparticle has only a mathematical origin: Weyl equation is not invariant with respect to space reflections.

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